Characterising Coupled Map Lattices

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Abstract

Here we investigate simple methods for characterising spatially extended, non-linear dynamical systems which can exhibit spatio-temporal chaotic behaviour. The systems considered are 2-D and 3-D square lattices of logistic iterators (maps), capable of nearest-neighbour interactions via Turing's diffusive coupling. These CMLs are generalisations of the well-known Cellular Automata. The exploration of the parameter space for such systems is computationally demanding. We propose a simple metric to characteristic the system's steady state(s) based on the Trace and individual values of the matrix of lattice states. This provides a robust descriptor for the multi-periodic 2-, 4-cycles, etc., which logistic CMLs exhibit, as does an isolated logistic iterator. A plot of the Trace versus the system parameters leads to a 3D generalisation of traditional bifurcation diagrams and provides rich illustrations of the system's dynamical behaviour. Such systems provide one route to understanding spatial chaos.

Introduction and background.

One of the simplest non-linear systems is that described by the logistic difference equation, or "mapping" [1]:

$$x[n+1] = c\, x[n]\, (1 - x[n]), \quad n = 0,1,2,.. \qquad (1)$$

Such equations have historical significance in studies of population dynamics, where they can be used to describe the growth of a species with limited resources. Extensions to competing species, predator-prey relations, etc. are well known. In recent decades Eqn(1) has found utility in the understanding of non-linear systems and chaos. The logistic system can display simple exponential decay or growth, oscillations, multiple steady states and, for $c > 3.8284$, chaotic behaviour.

An iterative relation, like Eqn(1), which transforms the current state x[n] to the next state x[n+1] is also called a mathematical 'map'. After multiple iterations, the system state may settle into one or more equilibrium values, $x^* = x[n \to \infty]$, called stable points, or the system may cycle through a finite sequence of such values, which can be called a n-cycle (eg. 2-cycle, 4-cycle, etc). These stable points and cycles are dependant on only the one parameter, c, the logistic constant. For a single, isolated logistic map, the basic cycle occurs for $1.0 < c < 3.0$ [1], with $x^* = 0$ for $c < 1$. As c increases the single stable point bifurcates into two stable states between which the systems switches indefinitely. At $c = 3.57$ it bifurcates again; and yet again at $c=3.6786$, becoming chaotic at $c = 3.8284$, a value close to Feigenbaum's constant. Eqn. 1 has non-trivial solutions only for $c < 4$. The plot of the values of the stable points, x^* vs. c is the well-known bifurcation diagram [1]. Eqn(1) is almost the simplest non-linear equation imaginable, yet already requires numerical solution which it repays with a rich variety of solutions and behaviour, including "deterministic chaos".

In this study we examine a spatially extended set of interacting logistic systems, each of which is located at a site on a regular (here square) lattice and whose dynamics is described by a map, such as in Eqn (1). Such a system could be applied, for example, to modelling populations of species interacting across geographically related zones –

eg. there might be a linear flux of population between adjacent sites. Extensions to irregular, ie. ‘amorphous’, or heteogeneous systems is anticipated. However, given the large and historical literature of analytic results for lattices [2], researchers are characterising the regular systems first.

The systems (maps) at the lattice sites are allowed to interact with their nearest neighbours (4 for a square 2D lattice) – thus the name, coupled map lattices (CML), as they have become known in the literature [3-4]. The logistic CML, which we study here, is one of the simplest cases. CMLs are a generalisation of the Celullar Automata (CA) systems that have been extensively studied over the past few decades [5] – in CAs, sites adopt binary values, rather than the continuously variable states allowed in CMLs. As an example, CA have found some application in the understanding of fluid turbulence [6]

More complicated maps (eg. the “tent” map [3,7-8]) have been studied and have provided insights into fundamental statistical mechanics [8] – such as the extension of equilibrium concepts, like the ensemble average, into the non-equilibrium domain. Egolf’s recent studies [8], for instance, have demonstrated the existence of a ‘coarse graining’ characteristic length scale, intermediate between molecular and macroscopic dimensions, such as those familiar in CFD.

Until recently, most studies of chaos concerned only time-dependent systems, or temporal chaos. As in statistics generally, little had been done to understand spatial chaos. Specific systems, such as the Lorenz equations for fluids and the Belousov-Zhabotinskii oscillating chemical reactions, provided early understanding of non-linear dynamics, and some beautiful results. CMLs provide a simple vehicle to explore more generalised chaotic behaviour in spatially extended systems. The presence of an additional parameter, which measures the coupling between adjacent maps, provides the opportunity to extend the usual 2-D bifurcation diagram to 3-D, as shown below.

The System

The first system used was a 2 dimensional, square lattice in which the state of each lattice site takes a real, continuous value in the range 0-1:

$$\underline{\boldsymbol{L}} = \{ x(i,j) \},\ \ i,j = 0, L-1;\ \ 0 < x < 1. \qquad (2)$$

Each value characterising the site state on the discrete lattice is x(i,j) where i denotes the i-th row, and j denotes the j-th column of a matrix, $\underline{\boldsymbol{L}}$. We investigated L = 9, 16, 20,40, 100, 256. Periodic boundary conditions are applied to eliminate edge effects, so that the mapping is applied on a topological torus. 3-D generalisations of (2) were investigated also. The lattice state is initialised by assigning a uniform random number to each site, so that $\langle x[n=0] \rangle = 0.5$. Each lattice site then evolves in time by an iteration based on the logistic function, as in Eqn. 1; ie f[x]=cx(1-x). Adjacent sites on lattice the interact via a constant, linear “diffusive coupling” [3,7] which dates back to Turing [9]. Thus the mapping is generalised to a lattice map, F[***L***,c,d] defined by:

$$x(i,j) \to (1-4d)f[x(i,j)] + d(f[x(i+1,j)]+f[x(i,j+1)]+f[x(i-1,j)] + f[x(i,j-1)])\ ;\quad i,j < L. \qquad (3)$$

Now this transform is dependent on d, the coupling constant, and c, the logistic constant which is incorporated in f. Thus the state value of a single site on this lattice is coupled with the adjacent states and its influence spreads across the lattice with successive iterations. A CML is somewhat reminiscent of the Ising model and has been investigated in mean-field approximations, etc. The coupling constant, d typically has been restricted to the range $0 < d < 0.25$ – previous studies indicate that the simple diffusively coupled CML exhibits spatial instabilities for $d > 0.492$ [7] and may give unbounded solutions. We followed the evolution of this CML by iterating the system until some steady state behaviour was achieved, typically for a minimum of 200 iterations, and for larger c and d, up to 1000 iterations.

Characterising the system

If we are to consider any properties of CML mapping, first we need a way to assign a characteristic "value" to a lattice of states. Following computational studies, described below, one of the metrics that we explored is based on the lattice's matrix representation. One of the simplest characteristics of a matrix is it's trace, ie. the sum of it's eigenvalues, which is also the sum of it's diagonal elements:

$$Tr = \sum x[i,i] \ , \quad i = 1,L \qquad (4)$$

For the lattice Trace to characterise the dynamical system in a unique way, we rely upon the empirical observation that the simulated system settles into a steady state or some mixture of steady states. There are several possible steady states: all lattice states x(i,j) are equal to some single value, x^* for constant c, in which the trace yields the same results expected from the traditional bifurcation diagram, these values are independent of the coupling constant, c as Eqn. 3 reduces to Eqn. 1. This is typically observed for logistic constant $c < 3$. For larger values of c, depending on the coupling constant, d each site state now cycles through 2 values, as does the Trace. These values are determined both by the logistic and coupling constants. For higher values of c, each site state cycles through a larger number of steady-state values, x^*_α, after an initial transient. At this point the Trace ceases to be a robust characterisation of the cyclic nature of the system. For a more robust characterisation of the system it is necessary to examine the manner in which the systems site states cycles through the x^* values. This finite set of values, that each site state adopts, is determined both by the logistic and coupling constants. This set is likely yield a good characterisation of the cyclic nature of the system. However this is not proven at this stage.

Results.

The CML system was investigated by computationally solving Eqn. 3 for a 2-D lattice of size L, with coupling, d in range (0,0.25) and c in (0,4): for c and d values outside these ranges, the lattice values, x(i,j) can become unrealistic, ie. outside the allowed range [0,1], or diverge. Initially, small systems, of size L = 9 and L = 16, were tested before simulating larger systems, with L = 16, 20,40, 100, 256. Each run, for given c and d, of the largest systems took 1-4 hours on a 400mHz sgi workstation. With 100s of runs typically needed to sample 800 c and 96 d values, it is clear that such spatial chaos studies are in the realm of HPC and can consume substantial compute resources. A simple, 3-D generalisation of the system was also studied, yielding similar results to those of the 2-D case. The dependence of the logistic system on the constant c is well known [1]. We examine below the behaviour of the system as the nearest-neighbour coupling strength, d also is varied. Figure 1, above illustrates a typical example of the

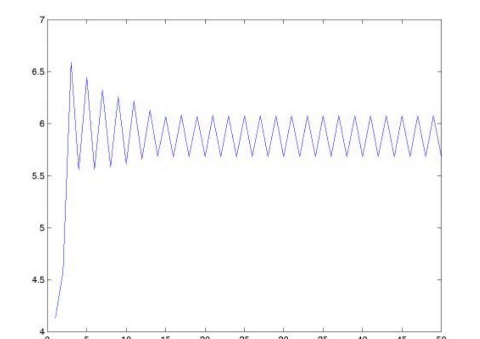


Fig. 1. Tr(n) for 2-cycle at c=3.3, d=0.01.

cycle in lattice trace, Tr over the first 50 iterations, for coupling constant, d= 0.01 and logistic constant, c=3.3 starting from a random 9x9 lattice. After an initial transient, the Tr values settle into a double valued steady state, where Tr switches between two values (approximately x* ~ 5.7/8 and 6.2/8) - this behaviour was followed for at least 200 iterations.

The pair of interchanging values is defined by the logistic and coupling constants, c and d. For a given lattice size, L the Trace (sum of the eigenvalues) oscillated between L*a and L*b, where a and b are constants (the x* values) dependant on c and d, but independent of L. Such a cycle occurred for multiple, separate randomly generated initial conditions of the lattice state. Thus this lattice adopts a slightly disordered checkerboard pattern, as shown in Fig. 2, which alternates in time. Here that lattices site values, in the range [0,1], are mapped to a colour (or grayscale).

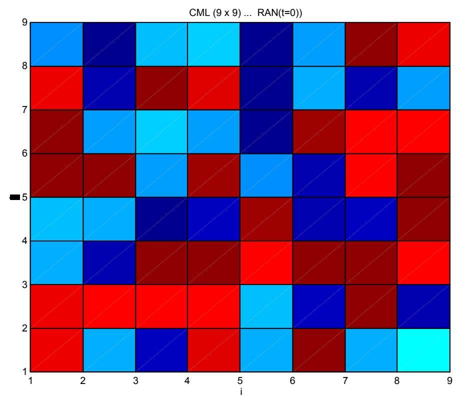


Fig. 2. Typical, instantaneous site pattern for the CML in Fig 1.

In order to understand why the Trace is an accurate representation of the systems lattice site values, we examine the spatial structure which the site values adopt. For a 2-cycle we denote values by a and b are constants (the x*). We observed that the steady state will eventually tend towards a set form; ie a variant of:

a b a b
b a b a ; for a 2-cycle where all x(i,j)= {a,b}
a b a b
b a b a ,

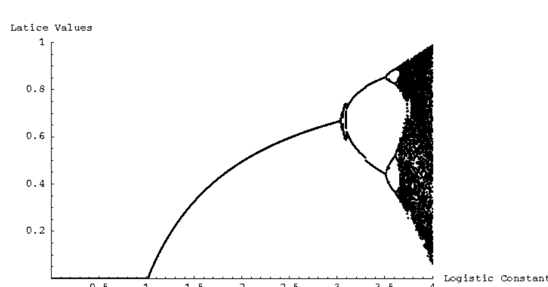
where the arrangement flips at each time step. This implies that a and b are the solutions to the two equations:

$$(1-4d)f[a] + 4df[b] = b$$
$$(1-4d)f[b] + 4df[a] = a \qquad (5)$$

for given coupling constant, d and logistic constant, c. The solutions to these equations were observed to describe the simulation results approximately (to within 1 part in 10^5 for small systems over 100s of iterations). For higher values of logistic constant c, again depending on the coupling constant d, it was observed that the site values settle into a 4-cycle, with x* values determined by both the logistic and coupling constant.

CML simulations were performed for multiple values of the coupling constant, d and logistic constant, c. The results resemble those found with a traditional bifurcation diagram, now extended to 3-D, with complex behaviour resulting from fixing d and varying c, and vice versa. The results are best illustrated by a constant-d cut through this generalised 3-D bifurcation diagram.

Figures 3-5 shows the simulation results of the L=40 system with increasing coupling constant, d = 0.0104, 0.0990 and 0.203,



respectively. The lattice value (Trace/L) is plotted vs. logisitic constant, c. For small coupling these resemble the familiar bifurcation diagrams for a single logisitic maps.

Fig. 3. Tr/L vs. c, for the CML with d = 0.01.

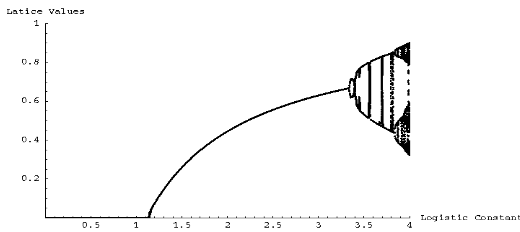


Fig. 4. Tr/L vs. c, for d = 0.0990.

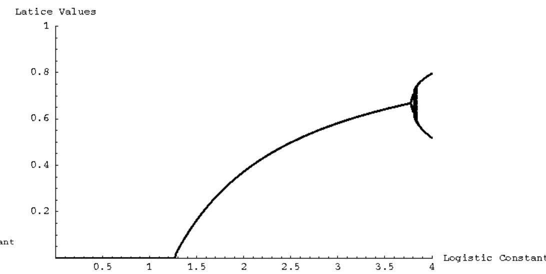


Fig. 5.Tr/L vs. c, for d = 0.203.

For d=0.01 the critical c for the 1- to 2-cycle transition has shifted slightly from c=3.0 to 3.05; and the 2- to 4-cycle transition occurs at c=3.5. By d=0.203, the critical c for the 1- to 2-cycle transition has increased from c=3.0 to c=3.80; a 4-cycle was not observed. The complexity of the simulation results mean that precise locations of higher order transitions could only be determined after more extensive study. Analytic solutions for the 2-cycle condition and for the stable points were obtained and found to be in agreement with the simulation results. Although there clearly exist higher order n-cycles (for n>2), an algebraic solution for that condition was unobtainable at this time. Fig. 3-5 show more complex behaviour for of these site values at large logistic constant, c and low values for the coupling constant, d. Interestingly, for increasing coupling (d > 0.2), and large c, the behaviour of this complex system begins to include new regularities; ie. the lattice appears to "rigidify", as suggested by Fig. 5, with intermittent localised cycling through the x* .

The general behaviour described above appears to be independent of the size of the system, L. However, we observed the presence of artefacts for certain values of L. Figure 6 shows simulation results for a L=100 system where d = 0.1. This is different from Fig. 4. These artefacts are due to the spatial layout which the lattice site values adopt

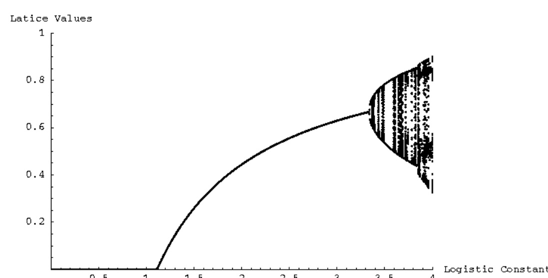
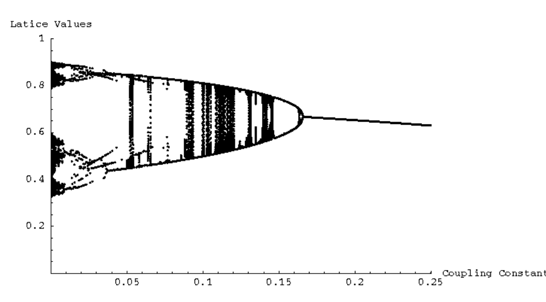


Fig. 6. Tr/L vs. c, for d = 0.1; L = 100

(cf. discussion preceding Eqn. 5). That is, a 2-cycle requires that L be divisible by at least 2 and the 4-cycles requires that L be divisible by 4, etc. Choosing large lattice sizes, L that are divisible by higher powers of 2 is likely to reduce such artefacts.

Now we consider the 3-D bifurcation diagram from the other perspective. Results obtained by fixing the logistic constant, c and varying the coupling constant, d (in Eqn. 3) clearly shows the dependence of the overall behaviour of the system on the coupling constant, d. Figures 7-8 shows a constant c-cut for L = 100 system.



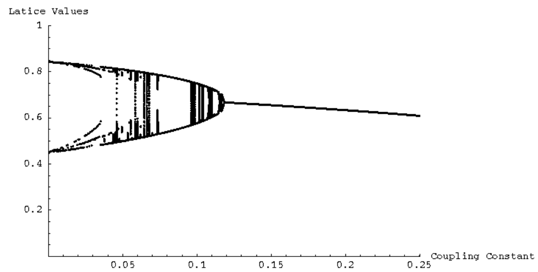


Fig. 7. Tr/L vs. d for c=3.41

Fig. 8. Tr/L vs. d for c=3.6

With values of L that are divisible by higher powers of 2 such as 256 and 1024, the structure on the bifurcation diagram is defined better. While these 2-D cuts give some sense of the system's behaviour, animated 3-D graphics are required to visualise and explore the rich structure of the full 3-D bifurcation diagram. A three dimensional slice-through of the L=40 system is available at our web-site[10]. Results for the 3-D cubic lattice will also be presented.

Conclusions.

Spatially extended non-linear systems modelled by a CML with linear, nearest-neighbour coupling are shown to exhibit regular, multi-periodic and chaotic dynamics. A new 3-D generalisation of the familiar 2D bifurcation diagram is demonstrated, based upon a matrix trace. The results extend familiar features of isolated logistic iterators. These coupled maps are expected to find applications in information processing and have already applied to statistics, for partitioning of data into clusters using inhomogeneous CMLs [12].

Acknowledgements.

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